

# Interest Rates

## Chapter 4

# Types of Rates

- Treasury rates
- LIBOR rates
- Repo rates

# Measuring Interest Rates

- The compounding frequency used for an interest rate is the unit of measurement
- The difference between quarterly and annual compounding is analogous to the difference between miles and kilometers

# Continuous Compounding

(Page 77)

- In the limit as we compound more and more frequently we obtain continuously compounded interest rates
- \$100 grows to  $\$100e^{RT}$  when invested at a continuously compounded rate  $R$  for time  $T$
- \$100 received at time  $T$  discounts to  $\$100e^{-RT}$  at time zero when the continuously compounded discount rate is  $R$

# Conversion Formulas

(Page 77)

Define

$R_c$ : continuously compounded rate

$R_m$ : same rate with compounding  $m$  times per year

$$R_c = m \ln \left( 1 + \frac{R_m}{m} \right)$$

$$R_m = m \left( e^{R_c/m} - 1 \right)$$

# Zero Rates

A zero rate (or spot rate), for maturity  $T$  is the rate of interest earned on an investment that provides a payoff only at time  $T$

# Example (Table 4.2, page 79)

| Maturity<br>(years) | Zero Rate<br>(% cont comp) |
|---------------------|----------------------------|
| 0.5                 | 5.0                        |
| 1.0                 | 5.8                        |
| 1.5                 | 6.4                        |
| 2.0                 | 6.8                        |

# Bond Pricing

- To calculate the cash price of a bond we discount each cash flow at the appropriate zero rate
- In our example, the theoretical price of a two-year bond providing a 6% coupon semiannually is

$$3e^{-0.05 \times 0.5} + 3e^{-0.058 \times 1.0} + 3e^{-0.064 \times 1.5} \\ + 103e^{-0.068 \times 2.0} = 98.39$$



# Bond Yield

- The bond yield is the discount rate that makes the present value of the cash flows on the bond equal to the market price of the bond
- Suppose that the market price of the bond in our example equals its theoretical price of 98.39
- The bond yield (continuously compounded) is given by solving

$$3e^{-y \times 0.5} + 3e^{-y \times 1.0} + 3e^{-y \times 1.5} + 103e^{-y \times 2.0} = 98.39$$

to get  $y=0.0676$  or 6.76%.

# Par Yield

- The par yield for a certain maturity is the coupon rate that causes the bond price to equal its face value.
- In our example we solve

$$\frac{C}{2}e^{-0.05 \times 0.5} + \frac{C}{2}e^{-0.058 \times 1.0} + \frac{C}{2}e^{-0.064 \times 1.5} + \left(100 + \frac{C}{2}\right)e^{-0.068 \times 2.0} = 100$$

to get  $C = 6.87$  (with a compound dig)

# Par Yield continued

In general if  $m$  is the number of coupon payments per year,  $P$  is the present value of \$1 received at maturity and  $A$  is the present value of an annuity of \$1 on each coupon date

$$C = \frac{(100 - 100P)m}{A}$$

# Sample Data (Table 4.3, page 80)

| Bond<br>Principal<br>(dollars) | Time to<br>Maturity<br>(years) | Annual<br>Coupon<br>(dollars) | Bond Cash<br>Price<br>(dollars) |
|--------------------------------|--------------------------------|-------------------------------|---------------------------------|
| 100                            | 0.25                           | 0                             | 97.5                            |
| 100                            | 0.50                           | 0                             | 94.9                            |
| 100                            | 1.00                           | 0                             | 90.0                            |
| 100                            | 1.50                           | 8                             | 96.0                            |
| 100                            | 2.00                           | 12                            | 101.6                           |

# The Bootstrap Method

- An amount 2.5 can be earned on 97.5 during 3 months.
- The 3-month rate is 4 times  $2.5/97.5$  or 10.256% with quarterly compounding
- This is 10.127% with continuous compounding
- Similarly the 6 month and 1 year rates are 10.469% and 10.536% with continuous compounding

# The Bootstrap Method continued

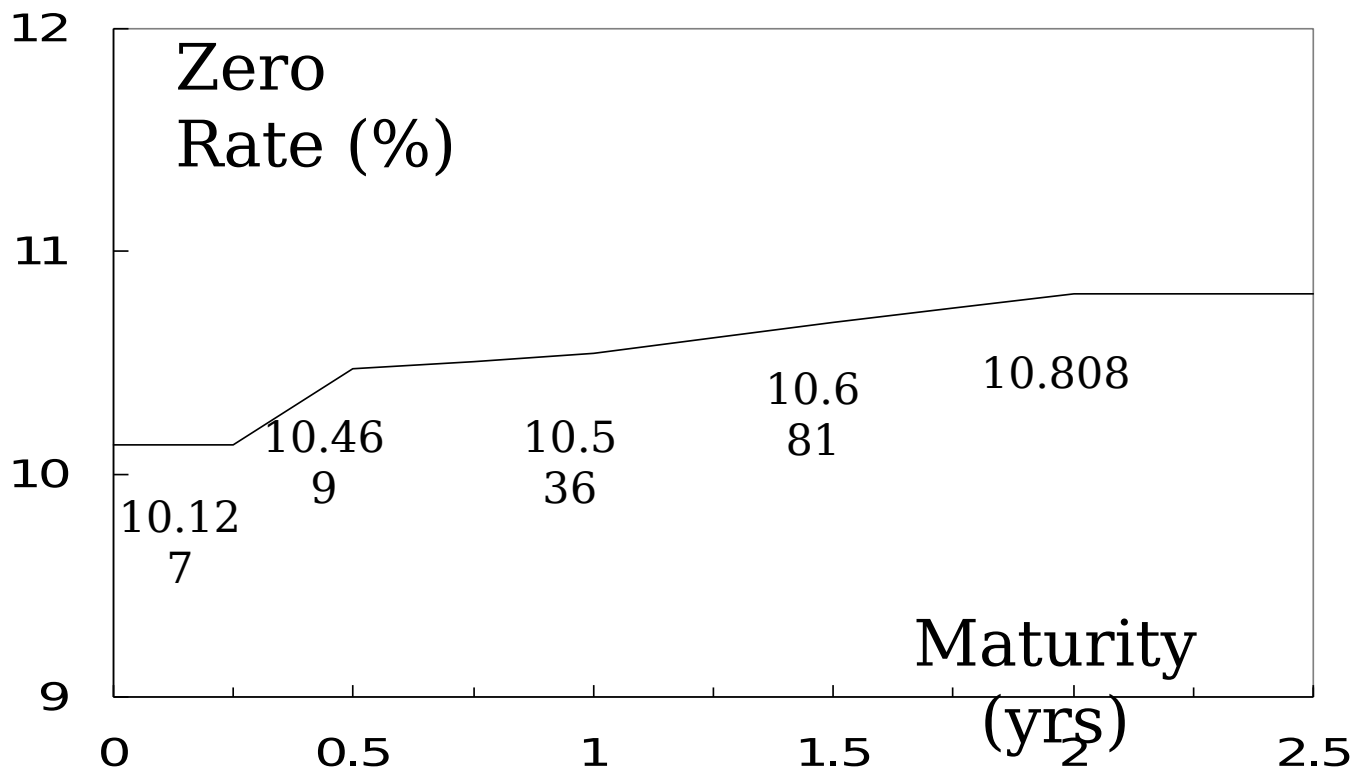
- To calculate the 1.5 year rate we solve

$$4e^{-0.10469 \times 0.5} + 4e^{-0.10536 \times 1.0} + 104e^{-R \times 1.5} = 96$$

to get  $R = 0.10681$  or 10.681%

- Similarly the two-year rate is 10.808%

# Zero Curve Calculated from the Data (Figure 4.1, page 82)



# Forward Rates

The forward rate is the future zero rate implied by today's term structure of interest rates



# Calculation of Forward Rates

Table 4.5, page 83

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| Year ( $n$ ) | $n$ -year<br>zero rate<br>(% per annum) | Forward Rate<br>for $n$ th Year<br>(% per annum) |
|--------------|-----------------------------------------|--------------------------------------------------|
| 1            | 3.0                                     |                                                  |
| 2            | 4.0                                     | 5.0                                              |
| 3            | 4.6                                     | 5.8                                              |
| 4            | 5.0                                     | 6.2                                              |
| 5            | 5.3                                     | 6.5                                              |

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# Formula for Forward Rates

- Suppose that the zero rates for time periods  $T_1$  and  $T_2$  are  $R_1$  and  $R_2$  with both rates continuously compounded.
- The forward rate for the period between times  $T_1$  and  $T_2$  is

$$\frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

# Instantaneous Forward Rate

- The instantaneous forward rate for a maturity  $T$  is the forward rate that applies for a very short time period starting at  $T$ . It is  $R_{+T} \frac{\partial R}{\partial T}$

where  $R$  is the  $T$ -year rate

# Upward vs Downward Sloping Yield Curve

- For an upward sloping yield curve:  
 $\text{Fwd Rate} > \text{Zero Rate} > \text{Par Yield}$
- For a downward sloping yield curve  
 $\text{Par Yield} > \text{Zero Rate} > \text{Fwd Rate}$

# Forward Rate Agreement

A forward rate agreement (FRA) is an agreement that a certain rate will apply to a certain principal during a certain future time period

# Forward Rate Agreement continued

- An FRA is equivalent to an agreement where interest at a predetermined rate,  $R_K$  is exchanged for interest at the market rate
- An FRA can be valued by assuming that the forward interest rate is certain to be realized

# Valuation Formulas (equations 4.9 and 4.10, pages 86-87)

- Value of FRA where a fixed rate  $R_K$  will be received on a principal  $L$  between times  $T_1$  and  $T_2$  is  $L(R_K - R_F)(T_2 - T_1)\exp(-R_2 T_2)$
- Value of FRA where a fixed rate is paid is  $L(R_F - R_K)(T_2 - T_1)\exp(-R_2 T_2)$
- $R_F$  is the forward rate for the period and  $R_2$  is the zero rate for maturity  $T_2$
- What compounding frequencies are used in these formulas for  $R_K$ ,  $R_F$ , and  $R_2$ ?

# Duration (page 87-88)

- Duration of a bond that provides cash flow  $c_i$  at time  $t_i$  is

$$D = \sum_{i=1}^n t_i \left[ \frac{c_i e^{-y t_i}}{B} \right]$$

where  $B$  is its price and  $y$  is its yield (continuously compounded)

- This leads to

$$\frac{\Delta B}{B} = - D \Delta y$$



# Duration Continued

- When the yield  $y$  is expressed with compounding  $m$  times per year

$$\Delta B = - \frac{BD\Delta y}{1 + y/m}$$

- The expression

$$\frac{D}{1 + y/m}$$

is referred to as the “modified duration”

# Convexity

The convexity of a bond is defined as

$$C = \frac{1}{B} \frac{\partial^2 B}{\partial y^2} = \frac{\sum_{i=1}^n c_i t_i^2 e^{-y t_i}}{B}$$

so that

$$\frac{\Delta B}{B} = -D \Delta y + \frac{1}{2} C (\Delta y)^2$$

# Theories of the Term Structure

Page 91-92

- Expectations Theory: forward rates equal expected future zero rates
- Market Segmentation: short, medium and long rates determined independently of each other
- Liquidity Preference Theory: forward rates higher than expected future zero rates

# Liquidity Preference Theory

- Suppose that the outlook for rates is flat and you have been offered the following choices

| Maturity | Deposit rate | Mortgage rate |
|----------|--------------|---------------|
| 1 year   | 3%           | 6%            |
| 5 year   | 3%           | 6%            |

- Which would you choose as a depositor? Which for your mortgage?

# Liquidity Preference Theory

## cont

- To match the maturities of borrowers and lenders a bank has to increase long rates above expected future short rates
- In our example the bank might offer

| Maturity | Deposit rate | Mortgage rate |
|----------|--------------|---------------|
| 1 year   | 3%           | 6%            |
| 5 year   | 4%           | 7%            |